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Thermopower as a possible probe of non-Abelian quasiparticle statistics in fractional quantum Hall liquids

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We show in this paper that thermopower is enhanced in non-Abelian quantum Hall liquids under appropriate conditions. This is because thermopower measures entropy per electron in the clean limit, while the degeneracy and entropy associated with non-Abelian quasiparticles enhance entropy when they are present. Thus thermopower can potentially probe non-Abelian nature of the quasiparticles, and measure their quantum dimension.

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Recently there has been very strong interest in unusual fractional quantum Hall (FQH) states,[1,2] whose quasiparticle excitations obey non-Abelian statistics.[3] Such interest is partially driven by the potential of using non-Abelian quasiparticles for quantum information storage and processing in an intrinsically fault-tolerant fashion.[3–7] At this time the most promising candidate for non-Abelian statistics is the FQH state at filling factor $\nu_0=5/2$,[8] in which the electrons in the half-filled first excited Landau level may condense into the Moore-Read (MR) (or Pfaffian) state[9] or its particle-hole conjugate (anti-Pfaffian state),[10,11] whose elementary quasiparticles have charge $e^*=e/4$. Theoretical support for the Pffafian or anti-Pfaffian state as an explanation for the FQH state at $\nu_0=5/2$ has come from a variety of numerical calculations.[12–17]

Phenomenologically, the 5/2 state looks very similar[8,18] to other FQH or integer quantum Hall states in ordinary transport measurements: one sees a quantized Hall resistance plateau and thermally activated longitudinal resistance. However, recent measurements, which involve tunneling between opposite edges across a constriction, have probed the quasiparticle charge e^* (Refs. 19 and 20) and may have also seen effects of non-Abelian statistics.[21] In this work we argue that bulk thermoelectric measurements, in particular thermopower, can also reveal the statistical properties of the non-Abelian quasiparticles under appropriate conditions. This is possible because thermopower can probe the entropy carried by non-Abelian quasiparticles, which is larger than that of Abelian quasiparticles at low temperature.

A key property of non-Abelian statistics is the appearance of ground-state degeneracy D that grows [up to an $O(1)$ prefactor] exponentially with the number of quasiparticles present in the system, N_q, when their positions are fixed:

$$D \sim d^{N_q}, \tag{1}$$

where $d>1$ is the quantum dimension[3] of the quasiparticle. For the non-Abelian quasiparticles in the MR Pfaffian or anti-Pfaffian state, $d=\sqrt{2}$. We will use them as the primary examples of our discussion below, although essentially all of our discussions apply to other non-Abelian FQH states. Such degeneracy results in a ground-state entropy

$$S_d = k_B \log D = k_B N_q \log d + O(1), \tag{2}$$

where k_B is the Boltzmann constant; i.e., each quasiparticle carries entropy $k_B \log d$. In principle, there exists very weak coupling among the quasiparticles that can lift the ground-state degeneracy.[22] However such coupling vanishes exponentially as a function of the distance between quasiparticles. Thus entropy formula (2) remains valid as long as the temperature T satisfies the condition

$$T_0 \ll T \ll T_1, \tag{3}$$

where $T_0 \sim \Delta e^{-l/l_0}$ (Δ is quasiparticle gap, l is the distance between quasiparticles, and l_0 is the characteristic size of the quasiparticle) is the temperature scale associated with quasiparticle couplings, and T_1 is the temperature scale associated with other (ordinary) excitations in the system, including those related to the quasiparticles' positional degrees of freedom. In principle, T_0 can be extremely low near the center of the quantum Hall plateau due to its exponential dependence on quasiparticle density, while T_1 should be larger. In particular, if the density of quasiparticles is sufficiently low, we expect that the quasiparticles will form a Wigner crystal due to the repulsion between quasiparticles, so the positional entropy should indeed disappear at low temperatures. We shall return to this issue later.

In a uniform system, the number of quasiparticles at low temperatures will be proportional to the deviation of the magnetic field B from the value B_0 at the center of the plateau, where the filling fraction is equal to the ideal value ν_0:

$$N_q = |(e/e^*)(B-B_0)/B_0| N_e, \tag{4}$$

where N_e is the number of electrons in the system. As a result the entropy $S_d=k_B N_q \log d$ grows linearly as B deviates from the center of the plateau B_0, within temperature range (3).

This entropy due to the presence of non-Abelian quasiparticles can be probed using thermopower. In a thermopower measurement, one sets up a temperature gradient ∇T, and voltage gradient $\mathbf{E}=-\nabla V$ is generated by the system to compensate for its effect so that no net electric current is flowing. The ratio between them,

$$Q = -\nabla V/\nabla T \tag{5}$$

is the thermopower (also known as the Seebeck coefficient). It is well known[23] that under suitable circumstances, Q measures the "entropy per charge carrier" in the system. This has been rigorously justified for electrons in a strong magnetic field in the *clean* limit, first for noninteracting electrons[24] and then for interacting electrons,[25] so

$$Q = -S/(eN_e). \tag{6}$$

In the following we present a derivation of Eq. (6) that is slightly simpler than but closely related to the arguments presented in Ref. 25. For an electron liquid without impurity scattering, the absence of net particle current requires that the variation in the liquid's internal pressure P balance with external potential ϕ:

$$\nabla P = \left(\frac{\partial P}{\partial \mu}\right)_T \nabla \mu + \left(\frac{\partial P}{\partial T}\right)_\mu \nabla T = -n \nabla \phi. \tag{7}$$

Here $n=N_e/A$ is electron number density, A is area, and μ is the local chemical potential measured from ϕ. The electrochemical potential is thus $\xi=\mu+\phi$, which is what an ideal voltage contact measures. From the grand potential relation

$$d\Omega = -SdT - PdA - N_e d\mu \tag{8}$$

follows the Maxwell relations $(\frac{\partial P}{\partial \mu})_{T,A}=(\frac{\partial N_e}{\partial A})_{T,\mu}=N_e/A=n$ and $(\frac{\partial P}{\partial T})_{\mu,A}=(\frac{\partial S}{\partial A})_{T,\mu}=S/A$. The last steps follow from the extensiveness of S, N_e, and A, which are proportional to each other when intensive quantities μ and T are fixed. Thus we find

$$n \nabla \mu + (S/A) \nabla T = -n \nabla \phi, \tag{9}$$

or

$$\nabla \xi / \nabla T = -S/N_e. \tag{10}$$

The voltage measured by voltmeter with ideal contacts is $\Delta\xi/q$, where q is the charge of the liquid's constituent particle, for electrons $q=-e$, while for holes $q=e$. Thus Eq. (6) follows for electron samples; for hole samples there is a corresponding sign change.

The simplicity of the argument above suggests that result (6) applies even in the *absence* of magnetic field, in the clean limit. We note that when studying thermoelectric effects, one usually starts with transport equations,[26] and thermopower is expressed as a ratio between transport coefficients.[25–27] In the absence of both disorder *and* magnetic field, transport coefficients are divergent and not well defined. However thermopower is still well defined and finite, and can be obtained easily using the hydrodynamic arguments presented above.

Strictly speaking, the hydrodynamic analysis above applies to a liquid whose internal stress tensor has only a diagonal component P. When the quasiparticles form a Wigner crystal, it may sustain some shear stress when driven out of equilibrium. This may result in correction to Eq. (7), which is proportional to the product of shear strain gradient (if present) and shear modulus of the crystal. However due to the long-range nature of the Coulomb interaction and the very small percentage of charge that actually forms the crystal, we expect the shear modulus to be much smaller than the bulk modulus, and such correction should be negligible.

Combining Eqs. (2), (4), and (6) we find within temperature window (3) and in the clean limit,

$$Q = -|(B-B_0)/B_0|(k_B/|e^*|)\log d. \tag{11}$$

Since $|e^*|$ can be measured independently,[19–21] Eq. (11) suggests that thermopower gives a direct measurement of quantum dimension d in the clean limit. It should be emphasized that it is $d>1$ that directly reveals the non-Abelian nature of the quasiparticle, while a fractional charge may correspond to either Abelian or non-Abelian quasiparticles. We note that in the low-temperature regime we are discussing here, phonons will be frozen out so that extrinsic effects such as phonon drag are absent. Thus thermopower should probe the intrinsic properties of the electron system.

We now turn the discussion to temperature range (3) within which our entropy formula (2) is valid. If the quasiparticles form a Wigner crystal, positional entropy comes from magnetophonons at low T, and one would expect $T_1 \approx T_D$, where T_D is the maximum phonon energy or Debye temperature. Treating the quasiparticles as point particles with charge e^* moving in the magnetic field B, they form a triangular lattice with lattice spacing

$$a = l_B\left[\frac{4\pi}{\sqrt{3}\nu_0}\frac{e}{e^*}\frac{B_0}{|B-B_0|}\right]^{1/2}, \tag{12}$$

where l_B is the magnetic length. Using the known magnetophonon spectrum of that system,[28] we obtain

$$k_B T_D \sim \frac{e^2}{\epsilon l_B}\sqrt{\frac{e}{|e^*|}}\left[\frac{\nu_0|B-B_0|}{B_0}\right]^{3/2}. \tag{13}$$

To justify treating quasiparticles as real particles for the specific case of $\nu_0=5/2$, we observe that they are vortices of a paired composite fermion superconductor. Using a duality transformation these vortices become particles, and the background composite fermion Cooper pairs become a magnetic field. While the short-range part of the quasiparticle interaction is not known, the long-range part is determined by the charge e^*, which is the most important in the low-density limit.

Another important temperature here is the melting temperature T_m. Its classical value is a small fraction of the Coulomb interaction energy between quasiparticles:

$$k_B T_m = \frac{1}{\Gamma}\frac{(e^*)^2}{\epsilon l_B}\left[\frac{\nu_0|B-B_0|}{2B_0}\frac{e}{|e^*|}\right]^{1/2}, \tag{14}$$

where $\Gamma \approx 137$.[29–31] Thus T_m and T_D have *different* dependences on $B-B_0$. This allows for the interesting possibility of $T_m < T_D$. If melting is continuous or very weakly first order, the liquid state that results from melting is expected to have strong short-range crystal order, and its positional entropy remains to be small compared to S_d as long as $T \ll T_D$. As a result we expect $T_m < T_1 < T_D$ in this case. On the other hand if melting is a strong first-order transition with latent heat of order $k_B T_m$ per quasiparticle, then we have $T_1 = T_m$.

For highest quality samples where the 5/2 FQH plateaus are observed, we typically have $B_0 \approx 4$ T, which results in $l_B \approx 100$ Å, and at the edge of plateau $|B-B_0|/B_0 \approx 1/200$, indicating that the quasiparticles form a (pinned) Wigner crystal up to that point, at low temperature. Using the dielectric constant $\epsilon = 13$ and $e^*/e = 1/4$, we obtain $T_m \approx 7$ mK and $T_D \approx 300$ mK at 5/2 plateau edge. We indeed have $T_m \ll T_D$ in this case.

To estimate T_0, we choose $l_0 = \sqrt{|e/e^*|} l_B$, which is the quasiparticle magnetic length, and $l = a$. Combining with $\Delta \approx 0.5$ K we obtain $T_0 \lesssim 1$ mK on the plateau. We note that these estimates are quite rough, especially that of T_0, due to the uncertainty in l_0 which enters the exponential.

In general, the presence of disorder will give corrections to result (6). In particular, a quasiparticle Wigner crystal is expected to be pinned by *weak* disorder in the linear-response regime, which is what gives rise to the FQH plateau in the first place. Pinning will also suppress its contribution to thermopower. Thus in order to observe the predicted effect on thermopower, one needs to depin the quasiparticles. The most straightforward way to do that is to melt the quasiparticle Wigner crystal by having $T > T_m$. To ensure positional entropy being small compared to S_d, we need $T \ll T_D$ *and* melting being a continuous or weak first-order transition. Experiment[29] as well as numerical simulation of classical Coulomb system suggests that this is indeed the case.[30–33] For $T \gtrsim T_m$, the liquid has strong short-range crystal order, and positional entropy can be estimated by summing the contributions from magnetophonons and free dislocations (which triggers melting in two dimensions). Just like in the crystal phase, the phonon contribution is small compared to S_d as long as $T \ll T_D$. The dislocation contribution

$$S_{\text{dis}} \approx N_{\text{dis}} \log(N_q/N_{\text{dis}}), \tag{15}$$

where N_{dis} is the number of free dislocations in the system. Thus $S_{\text{dis}} \ll S_d$ as long as $N_{\text{dis}} \ll N_q$. At low T we expect $N_{\text{dis}}/N_q \sim e^{-E_c/k_B T}$, where E_c is the dislocation core energy. Using results from a classical calculation at $T=0$,[34] one finds

$$E_c \approx 0.11 \frac{(e^*)^2}{\epsilon l_B} \left[\frac{\nu_0 |B-B_0|}{2\pi B_0} \frac{e}{|e^*|} \right]^{1/2}, \tag{16}$$

or $E_c/k_B T_m \approx 8$. One needs to take caution here, as both quantum and thermal fluctuations can renormalize E_c downward.[35]

While disorder cannot pin a quasiparticle liquid, it can still give rise to significant resistance as a liquid with a low density of dislocations tends to be very viscous. Thus in order to observe the non-Abelian entropy through Eq. (11), we need to work in the temperature range

$$T_m \lesssim T \ll T_D \tag{17}$$

and with sufficiently clean sample. The sample should be clean enough such that within the range of Eq. (17), the Hall resistivity ρ_{xy} is close to its classical value reached at high temperature, while the longitudinal resistivity ρ_{xx} is small compared to the quasiparticle contribution to ρ_{xy}.

Throughout our analysis, we have assumed that variations in ν due to inhomogeneities in the electron density are small compared to the average value of $|\nu-\nu_0|$, which puts additional stringent condition on sample quality. We have also assumed that there is no short-range attraction between quasiparticles strong enough to overcome their Coulomb repulsion and cause binding between pairs. If binding occurs, then quasiparticles might form a Wigner crystal of charge $e/2$ pairs, for small values of $|B-B_0|$, and the entropy S_d would be lost.

Another possible concern is that since non-Abelian degeneracy (1) is *not* associated with individual quasiparticles, but is a global property, the system might have difficulty accessing all the (nearly) degenerate states and the associated entropy at very low temperature. We do not believe this will be a problem in a thermopower measurement. Thermopower is driven, physically, by effects at the edges of the sample, where equilibrium is established between electrons in a lead or contact and quasiparticles within the two-dimensional (2D) quantized Hall system. This necessarily assumes that there is some reasonable rate of hopping of charge back and forth between the two-dimensional system and the leads, with creation and annihilation of quasiparticles close to the edge. As a result of this hopping, there should be a considerable amount of braiding in the edge region, which should give access to the full entropy of the states near the edges. We believe this is all that is required for the entropy to show up in a measurement of thermopower. However, we expect that even in the bulk, in the Wigner crystal phase, there will be significant braiding of quasiparticles on the laboratory time scale, due to motion of dislocations, interstitials, and vacancies. Moreover, even if one neglects braiding, splitting of the ground-state degeneracy due to the exponentially small interactions between quasiparticles will still correspond to a rate that is fast on a laboratory time scale, if one is not extremely close to the center of the plateau in a very uniform sample.

Thermopower has been studied in 2D electron gas in a magnetic field (especially in the quantum Hall regimes), both theoretically[25,36] and experimentally.[37,38] Experimentally it was found that Q reaches minima as a function of magnetic field on integer and fractional quantum Hall plateaus, and vanishes (apparently) exponentially as $T \to 0$ there. Thermopower is bigger at filling factors corresponding to compressible states, but still vanishes as $T \to 0$, typically in a power-law manner.[38] The central result of this work is that thermopower can be strongly enhanced near filling factors where a non-Abelian quantum Hall state is realized, and takes a roughly temperature-independent value within temperature range (17), which depends on the quantum dimension of the non-Abelian quasiparticle in sufficiently clean samples.

The mechanism for thermopower enhancement discussed here also applies to entropy generated by more conventional source of degeneracy, such as electron spin. Specific examples include the Wigner crystals formed on the integer quantum Hall plateaus around $\nu = 2n$, where n is an integer. In this case the quasiparticles are simply electrons or holes, and if the Landé g factor is tuned to be very close to zero by applying proper pressure, they each carry a spin entropy $k_B \log 2$ for temperature above the very small Zeeman splitting. As a result Eq. (11) applies in the appropriate temperature range, with $|e^*| = e$ and $d=2$. There are several advantages in attempting to observe the physics discussed here in these systems, as compared to the non-Abelian FQH states: (1) the gap is bigger and quantized plateau wider, allowing for a bigger field range for exploration; and (2) combined with bigger quasiparticle charge, this leads to higher T_D and T_m. These lead to a more accessible and possibly wider range of temperature for the validity of Eq. (11).

Note added. The present paper supersedes an earlier paper[39] by one of us on the same subject. Very recently a preprint[40] appeared in which the authors use ideas closely related to those discussed here to explore possibilities of probing non-Abelian entropy under equilibrium situations.

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can lift the non-Abelian degeneracy D. However as long as the rate of such braiding is small compared to $k_B T/\hbar$, the non-Abelian entropy S_d remains, which is most likely the case in the temperature range of interest [see Eq. (17)].

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